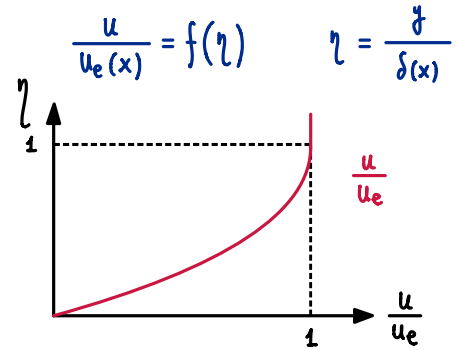


# Método de Pohlhausen

Si proponemos una cuártica para  $f(\eta)$ :

$$f(\eta) = C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3 + C_4\eta^4$$

5 constantes  $\rightarrow$  5 condiciones: 
$$\begin{cases} \eta = 0 : f|_0 = 0 ; f''|_0 = -\Lambda \\ \eta = 1 : f|_1 = 1 ; f'|_1 = f''|_1 = 0 \end{cases}$$



POHLHAUSEN SEPARA EL EFECTO DE LA CURVATURA EN LA PARED ( $\Lambda$ ) DEL VALOR DE  $f$  EN  $\eta = 1$  :  $f = f_1 + \Lambda f_2$

Descomposición:

$$f_1 = \sum_{i=0}^4 C_{1i} \eta^i$$

$$f_1 = C_{10} + C_{11}\eta + C_{12}\eta^2 + C_{13}\eta^3 + C_{14}\eta^4$$

$$f_1' = C_{11} + 2C_{12}\eta + 3C_{13}\eta^2 + 4C_{14}\eta^3$$

$$f_1'' = 2C_{12} + 6C_{13}\eta + 12C_{14}\eta^2$$

$\downarrow$

$$\eta = 0 : f_1 = 0 ; f_1'' = 0$$

$$\eta = 1 : f_1 = 1 ; f_1' = 0 ; f_1'' = 0$$

$$\eta = 0 : \begin{cases} C_{10} = 0 \\ C_{12} = 0 \end{cases}$$

$$\eta = 1 : \begin{cases} C_{11} + C_{13} + C_{14} = 1 \\ C_{11} + 3C_{13} + 4C_{14} = 0 \\ 6C_{13} + 12C_{14} = 0 \end{cases} \rightarrow \begin{cases} C_{11} = 2 \\ C_{13} = -2 \\ C_{14} = 1 \end{cases}$$

$$f_1 = 2\eta - 2\eta^3 + \eta^4$$

$$f_2 = \sum_{i=0}^4 C_{2i} \eta^i$$

$$f_2 = C_{20} + C_{21}\eta + C_{22}\eta^2 + C_{23}\eta^3 + C_{24}\eta^4$$

$$f_2' = C_{21} + 2C_{22}\eta + 3C_{23}\eta^2 + 4C_{24}\eta^3$$

$$f_2'' = 2C_{22} + 6C_{23}\eta + 12C_{24}\eta^2$$

$\downarrow$

$$\eta = 0 : f_2 = 0 ; f_2'' = -1$$

$$\eta = 1 : f_2 = 0 ; f_2' = 0 ; f_2'' = 0$$

$$\eta = 0 : \begin{cases} C_{20} = 0 \\ C_{22} = -\frac{1}{2} \end{cases}$$

$$\eta = 1 : \begin{cases} C_{21} + C_{23} + C_{24} = \frac{1}{2} \\ C_{21} + 3C_{23} + 4C_{24} = 1 \\ 6C_{23} + 12C_{24} = 1 \end{cases} \rightarrow \begin{cases} C_{21} = \frac{1}{6} \\ C_{23} = \frac{1}{2} \\ C_{24} = -\frac{1}{6} \end{cases}$$

$$f_2 = \frac{1}{6}\eta(1-\eta)^3$$

Zona externa de la capa límite :

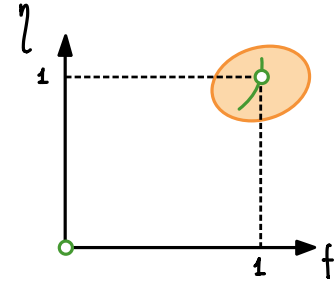
$$1 - \eta = \epsilon \quad (\epsilon \ll 1)$$

$$f_1 = 2\eta - 2\eta^3 + \eta^4 = 2(1-\epsilon) - 2(1-\epsilon)^3 + (1-\epsilon)^4 = 2 - 2\epsilon -$$

$$- 2(1 - 3\epsilon + 3\epsilon^2 - \epsilon^3) + (1 - 2\epsilon + \epsilon^2) = 2 - 2\epsilon - 2 + 6\epsilon - 6\epsilon^2 +$$

$$+ 2\epsilon^3 + 1 - 4\epsilon + 6\epsilon^2 - 4\epsilon^3 + \epsilon^4 = 1 - 2\epsilon^3 + \epsilon^4 \longrightarrow f_1 \approx 1 + O(\epsilon^3)$$

$$f_2 = \frac{1}{6}\eta(1-\eta)^3 = \frac{1}{6}(1-\epsilon)[1-(1-\epsilon)]^3 = \frac{1}{6}(1-\epsilon)\epsilon^3 = \frac{1}{6}\epsilon^3 - \frac{1}{6}\epsilon^4 \longrightarrow f_2 \approx \frac{\epsilon^3}{6} + O(\epsilon^4)$$



$$\longrightarrow \frac{f_2}{f_1} \approx \frac{\epsilon^2}{6} \ll 1$$

EL EFECTO INTRODUCIDO POR LA CURVATURA NO AFECTA CASI EN ESTA ZONA  $\longrightarrow$

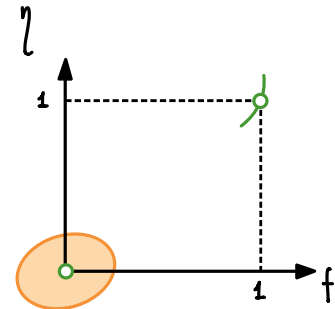
$\longrightarrow$  EN POHLHAUSEN LA REGIÓN EXTERIOR DE LA CAPA LÍMITE ES POCO SENSIBLE AL GRADIENTE DE PRESIONES, LO QUE CONLLEVARÁ CIERTO ERROR EN LOS RESULTADOS.

Cerca de la pared :

$$\eta = \epsilon \quad (\epsilon \ll 1)$$

$$f_1 = 2\eta - 2\eta^3 + \eta^4 = 2\epsilon - 2\epsilon^3 + \epsilon^4 \longrightarrow f_1 \approx 2\epsilon + O(\epsilon^3)$$

$$f_2 = \frac{1}{6}\eta(1-\eta)^3 = \frac{1}{6}(1-3\epsilon+3\epsilon^2-\epsilon^3) = \frac{\epsilon}{6} - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{2} - \frac{\epsilon^4}{6} \longrightarrow f_2 \approx \frac{\epsilon}{6} + O(\epsilon^2)$$



$$\frac{f_2}{f_1} \approx \frac{1}{12}$$

Cerca de la pared, para que el efecto de  $\Lambda$  sea apreciable, debe ser  $\Lambda \sim O(10)$  : VARIACIONES IMPORTANTES CON RESPECTO A  $\Lambda = 0$  (Blasius).

Variaciones de orden unidad tendrían poca influencia cerca de la pared.

Procedemos a obtener  $\delta_1$ ,  $\delta_2$ ,  $H_{12}$  y  $C_f$ .

Espesor de desplazamiento :

$$\delta_1 = \delta \int_0^1 (1-f) d\eta = \delta \int_0^1 \left(1 - f_1 - \Lambda f_2\right) d\eta \longrightarrow \frac{\delta_1}{\delta} = \int_0^1 \left(1 - 2\eta + 2\eta^3 - \eta^4 - \frac{\Lambda}{6}\eta + \frac{\Lambda}{2}\eta^2 - \frac{\Lambda}{2}\eta^3 + \frac{\Lambda}{6}\eta^4\right) d\eta = \int_0^1 \left[1 - \left(2 + \frac{\Lambda}{6}\right)\eta + \frac{\Lambda}{2}\eta^2 + \left(2 - \frac{\Lambda}{2}\right)\eta^3 - \left(1 - \frac{\Lambda}{6}\right)\eta^4\right] d\eta =$$

$$= \eta \Big|_0^1 - \frac{2 + \frac{\Lambda}{6}}{2} \eta^2 \Big|_0^1 + \frac{\Lambda}{6} \eta^3 \Big|_0^1 + \frac{2 - \frac{\Lambda}{2}}{4} \eta^4 \Big|_0^1 - \frac{1 - \frac{\Lambda}{6}}{5} \eta^5 \Big|_0^1 = 1 - 1 - \frac{\Lambda}{12} + \frac{\Lambda}{6} + \frac{1}{2} -$$

$$- \frac{\Lambda}{8} - \frac{1}{5} + \frac{\Lambda}{30} = \frac{3}{10} - \frac{\Lambda}{120} \rightarrow \boxed{\frac{\delta_1}{\delta} = \frac{1}{10} \left( 3 - \frac{\Lambda}{12} \right)}$$

Espesor de cantidad de movimiento :

$$\delta_2 = \delta \int_0^1 f(1-f) d\eta \rightarrow \frac{\delta_2}{\delta} = \underbrace{\int_0^1 f d\eta}_I - \underbrace{\int_0^1 f^2 d\eta}_J$$

Resolvemos I :

$$I = \int_0^1 f d\eta = \int_0^1 d\eta - \int_0^1 (1-f) d\eta = 1 - \frac{1}{10} \left( 3 - \frac{\Lambda}{12} \right) \rightarrow \boxed{I = \frac{1}{10} \left( 7 + \frac{\Lambda}{12} \right)}$$

Resolvemos J :

$$J = \int_0^1 f^2 d\eta = \int_0^1 (f_1 + \Lambda f_2)^2 d\eta = \int_0^1 (f_1^2 + 2\Lambda f_1 f_2 + \Lambda^2 f_2^2) d\eta = \overbrace{\int_0^1 f_1^2 d\eta}^{J_1} + 2\Lambda \overbrace{\int_0^1 f_1 f_2 d\eta}^{J_2} + \Lambda^2 \overbrace{\int_0^1 f_2^2 d\eta}^{J_3}$$

$$J_1 = \int_0^1 f_1^2 d\eta = \int_0^1 (2\eta - 2\eta^3 + \eta^4)^2 d\eta = \int_0^1 (4\eta^2 - 4\eta^4 + 2\eta^5 - 4\eta^4 + 4\eta^6 - 2\eta^7 + 2\eta^5 - 2\eta^7 + \eta^8) d\eta = \frac{4}{3} - \frac{4}{5} + \frac{1}{3} - \frac{4}{5} + \frac{4}{7} - \frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{9} = \frac{367}{630} \rightarrow \boxed{J_1 = \frac{367}{630}}$$

$$J_2 = 2\Lambda \int_0^1 f_1 f_2 d\eta = 2\Lambda \int_0^1 (2\eta - 2\eta^3 + \eta^4) \frac{1}{6} \eta (1-\eta)^3 d\eta \xrightarrow{\text{CASIO}} \boxed{J_2 = \frac{71}{7560} \Lambda}$$

$$J_3 = \Lambda^2 \int_0^1 f_2^2 d\eta = \Lambda^2 \int_0^1 \frac{1}{36} \eta^2 (1-\eta)^6 d\eta = \xrightarrow{\text{CASIO}} \boxed{J_3 = \frac{1}{9072} \Lambda^2}$$

Por tanto :

$$\frac{\delta_2}{\delta} = \frac{7}{10} + \frac{\Lambda}{120} - \frac{367}{630} - \frac{71}{7560} \Lambda - \frac{1}{9072} \Lambda^2 = \frac{37}{315} - \frac{4}{315} \frac{\Lambda}{12} - \frac{1}{63} \left( \frac{\Lambda}{12} \right)^2 \rightarrow$$

$$\rightarrow \boxed{\frac{\delta_2}{\delta} = \frac{1}{63} \left[ \frac{37}{5} - \frac{4}{5} \frac{\Lambda}{12} - \left( \frac{\Lambda}{12} \right)^2 \right]}$$

Factor de forma:

$$H_{12} = \frac{\delta_1}{\delta_2} = \frac{\frac{\delta_1}{\delta}}{\frac{\delta_2}{\delta}} = \frac{\frac{1}{10} \left( 3 - \frac{\Lambda}{12} \right)}{\frac{1}{63} \left[ \frac{37}{5} - \frac{4}{5} \frac{\Lambda}{12} - \left( \frac{\Lambda}{12} \right)^2 \right]} \rightarrow H_{12} = \frac{63}{10} \frac{3 - \frac{\Lambda}{12}}{\frac{37}{5} - \frac{4}{5} \frac{\Lambda}{12} - \left( \frac{\Lambda}{12} \right)^2}$$

Coefficiente de fricción:

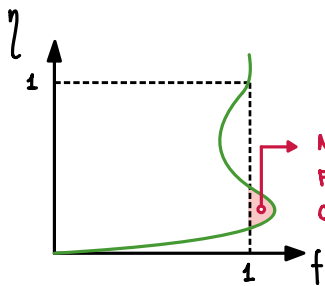
$$\frac{C_f}{2} = \frac{\nu}{u_e \delta} \left. \frac{df}{d\eta} \right|_{\eta=0} = \frac{\nu}{u_e \delta} \frac{d}{d\eta} \left[ 2\eta - 2\eta^3 + \eta^4 + \Lambda \frac{1}{6} \eta (1-\eta)^3 \right]_{\eta=0} = \frac{\nu}{u_e \delta} \left[ 2 - 6\eta + \right.$$

$$\left. + 4\eta^3 - \frac{\Lambda}{2} \eta (1-\eta)^2 + \frac{\Lambda}{6} (1-\eta)^3 \right]_{\eta=0} = \frac{\nu}{u_e \delta} \left( 2 + \frac{\Lambda}{6} \right) \rightarrow \frac{C_f}{2} = \frac{2\nu}{u_e \delta} \left( 1 + \frac{\Lambda}{12} \right)$$

Si  $\Lambda = -12$ :  $C_f = 0$  (SEPARACIÓN EN POHLHAUSEN)  $\rightarrow \Lambda \geq -12$  CONDICIÓN NECESARIA 

Pohlhausen no sirve si  $\Lambda < -12$  porque implicaría curvatura negativa de la pared  $\rightarrow$  flujo reverso

Para  $\Lambda > 12 \exists \eta^*/f(\eta^*) > 1$ :



$$\rightarrow \Lambda \leq 12 \rightarrow -12 \leq \Lambda \leq 12 \rightarrow -1 \leq \frac{\Lambda}{12} \leq 1$$

Hay otro parámetro interesante:

$$\lambda = \frac{\delta_2^2}{\nu} \frac{du_e}{dx}$$

$$-0'1567 \leq \lambda \leq 0'0948$$

$$\lambda = \underbrace{\left( \frac{\delta_2}{\delta} \right)^2}_{\Lambda} \frac{du_e}{dx} \frac{\delta^2}{\nu} = \frac{\Lambda}{3969} \left[ \frac{37}{5} - \frac{4}{5} \frac{\Lambda}{12} - \left( \frac{\Lambda}{12} \right)^2 \right]^2 \rightarrow \lambda = \frac{\Lambda}{3969} \left[ \frac{37}{5} - \frac{4}{5} \frac{\Lambda}{12} - \left( \frac{\Lambda}{12} \right)^2 \right]^2$$

## VALIDACIÓN CON BLASIUS

Vamos a ver cómo de bueno es este método comparado con la solución de Blasius:

Para una placa plana sin gradiente de presiones:  $\Lambda = 0 \rightarrow f = f_1 = 2\eta - 2\eta^3 + \eta^4$ ,  
y entonces:

$$\delta_1 = \frac{3}{10} \delta \quad ; \quad \delta_2 = \frac{37}{315} \delta \quad ; \quad \frac{C_f}{2} = \frac{2\nu}{u_e \delta}$$

Ecuación de Karman:

$$\frac{d\delta_2}{dx} + \underbrace{(2 + H_{12}) \frac{1}{u_e} \frac{du_e}{dx}}_0 \delta_2 = \frac{C_f}{2} \rightarrow \frac{37}{315} \frac{d\delta}{dx} = \frac{2\sqrt{v}}{u_e \delta} \rightarrow \int_0^\delta \tilde{\delta} d\tilde{\delta} = \int_0^x \left[ \frac{630}{37} \frac{\sqrt{v}}{u_e} \right] dx \xrightarrow{\delta(0)=0}$$

$$\rightarrow \frac{\delta^2}{2} = \frac{630}{37} \frac{\sqrt{v}}{u_e} x \rightarrow \frac{\delta}{x} = 6 \sqrt{\frac{35}{37}} Re_x^{-1/2}$$

Con  $\frac{\delta}{x}$  podemos obtener  $\frac{\delta_1}{x}$ ,  $\frac{\delta_2}{x}$  y  $\frac{C_f}{2}$ :

$$\frac{\delta_1}{x} = \frac{3}{10} \frac{\delta}{x}$$

$$\frac{\delta_1}{x} = \frac{9}{5} \sqrt{\frac{35}{37}} Re_x^{-1/2} \approx 1.751 Re_x^{-1/2}$$

En la solución exacta de Blasius era  $1.721 Re_x^{-1/2} \rightarrow 1.7\%$  de error

$$\frac{\delta_2}{x} = \frac{37}{315} \frac{\delta}{x}$$

$$\frac{\delta_2}{x} = \frac{74}{105} \sqrt{\frac{35}{37}} Re_x^{-1/2} \approx 0.685 Re_x^{-1/2}$$

En la solución exacta de Blasius era  $0.664 Re_x^{-1/2} \rightarrow 3.2\%$  de error

Karman

$$\frac{C_f}{2} = \frac{d\delta_2}{dx} = \frac{d}{dx} \left( \frac{74}{105} \sqrt{\frac{35}{37}} \sqrt{\frac{v x}{u_e}} \right) = \frac{74}{105} \sqrt{\frac{35}{37}} \sqrt{\frac{v}{u_e}} \frac{1}{2\sqrt{x}}$$

$$\frac{C_f}{2} = \frac{74}{105} \sqrt{\frac{35}{37}} Re_x^{-1/2} \approx 0.685 Re_x^{-1/2}$$

Blasius:  $0.664 Re_x^{-1/2}$  (3.2% error)

A la vista de los errores, Pohlhausen es una buena aproximación de la solución de Blasius.

## VALIDACIÓN CON FALKNER - SKAN

Por último, vamos a ver cómo de bueno es este método para capa límite con gradiente de presiones.

Ecuación de Karman:

$$\frac{d\delta_2}{dx} + (2 + H_{12}) \frac{1}{u_e} \frac{du_e}{dx} \delta_2 = \frac{C_f}{2} \xrightarrow{\cdot \frac{u_e \delta_2}{\sqrt{v}}} \frac{u_e \delta_2}{\sqrt{v}} \frac{d\delta_2}{dx} + (2 + H_{12}) \lambda = \frac{C_f}{2} \frac{u_e}{\sqrt{v}} \delta_2 \xrightarrow{\frac{C_f}{2} = \frac{2\sqrt{v}}{u_e \delta} \left(1 + \frac{\Lambda}{12}\right)}$$

$$\rightarrow \frac{u_e}{\sqrt{v}} \frac{d\left(\frac{\delta_2^2}{2}\right)}{dx} + (2 + H_{12}) \lambda = \underbrace{\frac{\delta_2}{\delta} \cdot 2 \left(1 + \frac{\Lambda}{12}\right)}_{T(\lambda)} \xrightarrow{\frac{\delta_2^2}{\sqrt{v}} = \frac{\lambda}{du_e/dx}} \underbrace{u_e \frac{d}{dx} \left(\frac{\lambda}{du_e/dx}\right)}_{F(\lambda)} = 2 \left[ T(\lambda) - (2 + H_{12}) \lambda \right]$$

Por tanto:

$$u_e \frac{d}{dx} \left( \frac{\lambda}{du_e/dx} \right) = F(\lambda) = 2 \left[ T(\lambda) - (2 + H_{12}) \lambda \right]$$

NECESITA UNA CONDICIÓN INICIAL, QUE PUEDE SER  $\delta$ ,  $\delta_1$ ,  $\delta_2$  o  $\Lambda$  EN EL PUNTO INICIAL DE INTEGRACIÓN

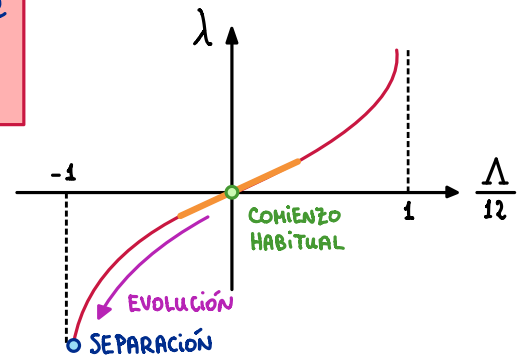
$$T(\lambda) = 2 \frac{\delta_2}{\delta} \left(1 + \frac{\Lambda}{12}\right) = \frac{C_f}{2} \frac{u_e \delta_2}{\sqrt{v}}$$

$$\frac{C_f}{2} = \frac{2\sqrt{v}}{u_e \delta} \left(1 + \frac{\Lambda}{12}\right)$$

Dado que  $\frac{\lambda}{\Lambda} = \frac{\delta}{\delta_2} \rightarrow \lambda = \frac{\Lambda}{63^2} \left[ \frac{37}{5} - \frac{4}{5} \frac{\Lambda}{12} - \left( \frac{\Lambda}{12} \right)^2 \right]^2$

Si tenemos una aceleración/deceleración moderada ( $\Lambda \sim 1$ ), la

relación entre  $\lambda$  y  $\Lambda$  se hace lineal:  $\lambda \approx \left( \frac{37}{315} \right)^2 \Lambda$



Al igual que analizamos la separación, también analizamos la presencia de un punto de remanso ( $u_e = 0$ ).

Si  $u_e = 0$  no podemos arrancar la solución, y para solucionarlo recurrimos a la solución de Falkner - Skan :

$$u_e = Ax^m \quad (m > 0) \rightarrow \frac{du_e}{dx} = mA x^{m-1} \rightarrow \frac{1}{du_e/dx} = \frac{1}{m} \frac{x^{1-m}}{A}$$

$$Ax^m \frac{d}{dx} \left( \frac{\lambda}{mA} x^{1-m} \right) = F(\lambda)$$

$$x^m \frac{d}{dx} (\lambda x^{1-m}) = m F(\lambda)$$

$$x^m \frac{d\lambda}{dx} x^{1-m} + x^m \lambda (1-m) x^{-m} = m F(\lambda)$$

$$x \frac{d\lambda}{dx} + \lambda (1-m) = m F(\lambda)$$

LA EVOLUCIÓN DE  $\lambda$  SIGUE ESTA ECUACIÓN DIFERENCIAL

### NOTA

$$\lambda = \frac{\delta_2^2}{\nu} \frac{du_e}{dx}$$

Falkner - Skan :

$$\begin{cases} \frac{\delta_2}{x} = cte(m) \cdot Re_x^{-1/2} \\ u_e = Ax^m ; \frac{du_e}{dx} = \frac{m u_e}{x} \end{cases}$$

$$\lambda = \frac{cte(m) x^2}{\nu Re_x} \frac{du_e}{dx} = cte(m) \frac{x^2}{\nu \frac{u_e x}{x}} \frac{m u_e}{x} = m cte(m) \rightarrow \lambda = cte(m)$$

POHLHAUSEN ES CONSISTENTE CON FALKNER - SKAN

$$\lambda = cte \rightarrow \frac{d\lambda}{dx} = 0 \rightarrow \lambda_{P-PR} (1-m) = m F(\lambda_{P-PR}) \rightarrow \lambda_{P-PR}$$

↗ POHLHAUSEN  
↘ PUNTO DE REMANSO (FALKER - SKAN)

$$\begin{cases} x \frac{d\lambda}{dx} + \lambda (1-m) = m F(\lambda) \\ x = x_0 : \lambda = \lambda_{P-PR} \end{cases}$$

AHORA SÍ PODEMOS ARRANCAR LA SOLUCIÓN si  $u_e = 0$

¡OJO! ESTAR CERCA DEL PR ( $u_e = 0$ ) NO ES LO MISMO

QUE FLUJO DE PR ( $u_e = Ax$ ).

$$\frac{\Lambda}{12} = -1$$

En cuanto a la separación :  $\lambda_{s-p} = -0'1567$  (Pohlhausen) ;  $\lambda_{s-fs} = -0'067$  (Falkner - Skan)

EL ERROR ES MUY GRANDE EN LA ZONA DE SEPARACIÓN PORQUE EL MÉTODO DE POHLHAUSEN ES POCO SENSIBLE AL GRADIENTE DE PRESIONES DEL FLUJO EXTERIOR (DEMASIADO OPTIMISTA)

Tabla resumen de resultados :

$\frac{\Lambda}{12}$	$\lambda_p$	$\lambda_{fs}$	$T(\lambda)$	$H_{12}(\lambda)$	$F(\lambda)$
-1	-0'1567	-0'067	0	3'5	1'72
0	0	0	0'2349	2'55	0'47
1	0'0948	<del>0</del>	0'3556	2'25	-0'0948

SEPARACIÓN POHLHAUSEN

BLASIUS

MÁXIMA ACELERACIÓN COMPATIBLE CON QUE NO HAYA SOBREVELOCIDADES DENTRO DE LA CAPA LÍMITE.

NO CAMBIA DEMASIADO, ESPECIALMENTE EN LA ZONA DE ACELERACIÓN ( $\frac{dU_e}{dx} > 0$ , es decir,  $\lambda > 0$ ).